

# Smoothed-TV Regularization for Hölder Continuous Functions

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## Abstract.

This work aims to explore the regularity properties of the smoothed-TV regularization for the functions is of the class Hölder continuous. Over some compact and convex domain  $\Omega$ , we study construction of multivariate function  $\varphi(\mathbf{x}) : \Omega \subset \mathbb{R}^3 \rightarrow \mathbb{R}_+$  as the optimized solution to the following convex minimization problem

$$\arg \min_{\Omega} \left\{ F_{\alpha}(\cdot, f^{\delta}) := \frac{1}{2} \|\mathcal{T}(\cdot) - f^{\delta}\|_{\mathcal{H}}^2 + \alpha J(\cdot) \right\},$$

where the penalizer  $J(\cdot) : \mathcal{C}^1(\Omega, \mathbb{R}^3) \rightarrow \mathbb{R}_+$  is the smoothed total variation penalizer

$$J(\cdot) = \int_{\Omega} \sqrt{\|\nabla(\cdot)\|_2^2 + \beta} d\mathbf{x},$$

for a fixed  $0 < \beta < 1$ . We assume our target function to be Hölder continuous. With this assumption, we establish relation between total variation of our target function and its Hölder coefficient. We prove that the smoothed-TV regularization is an admissible regularization strategy by evaluating the discrepancy  $\|\mathcal{T}\varphi_{\alpha} - f^{\delta}\| \leq \tau\delta$  for some fixed  $\tau \geq 1$ . To do so, we need to assume that the target function to be class of  $\mathcal{C}^{1+}(\Omega)$ . From here, under the fact that the penalty  $J(\cdot)$  is strongly convex, we move on to showing the convergence of  $\|\varphi_{\alpha} - \varphi^{\dagger}\|$ , for  $\varphi_{\alpha}$  is the optimum and  $\varphi^{\dagger}$  is the true solution for the given minimization problem above. We demonstrate that strong convexity and 2-convexity are actually different names for the same concept. In addition to these facts, we make use of Bregman divergence in order to be able to quantify the rate of convergence.

**Keywords.** Hölder continuity, Bounded variation, smoothed total variation, Morozov discrepancy.

## 1. Introduction

As alternative to well established Tikhonov regularization, [26, 27], studying convex variational regularization with any penalizer  $J(\cdot)$  has become important over the last decade. Introducing a new image denoising method named as *total variation*, [28], is commencement of this study. Application and analysis of the method have been widely carried out in the communities of inverse problems and optimization,

[1, 3, 4, 8, 9, 10, 14, 15, 31]. Particularly, formulating the minimization problem as variational problem and estimating convergence rates with variational source conditions has also become popular recently, [7, 18, 19, 20, 25]. Unlike in the available literature, we define discrepancy principle for the smoothed-TV regularization under a particular rule for the choice of regularization parameter. Furthermore, still with the same regularization parameter, we manage to show that smoothed-TV regularization is an admissible regularization strategy with Hölder continuity.

We are tasked with constructing the regularized solution  $\varphi_\alpha$  over some compact and convex domain  $\Omega \subset \mathcal{H}$ , for the following variational minimization problem,

$$\varphi_{\alpha(\delta)} \in \arg \min_{\varphi \in \mathcal{H}} \left\{ F_\alpha(\varphi, f^\delta) := \frac{1}{2} \|\mathcal{T}\varphi - f^\delta\|_{\mathcal{H}}^2 + \alpha J(\varphi) \right\}, \quad (1.1)$$

for the penalty term  $J(\varphi) : \mathcal{C}^2(\Omega, \mathcal{H}) \rightarrow \mathbb{R}_+$  defined by

$$J(\varphi) = \int_{\Omega} \sqrt{\|\nabla \varphi\|_2^2 + \beta} d\mathbf{x}, \quad (1.2)$$

and  $\alpha > 0$  is the regularization parameter. It is expected that the perturbed given data is  $f^\delta \notin \mathcal{R}(\mathcal{T})$  lies in some  $\delta$ -ball  $\mathcal{B}_\delta(f^\dagger)$  centered at the true data  $f^\dagger$ , *i.e.*  $\|f^\dagger - f^\delta\| \leq \delta$ . The compact forward operator  $\mathcal{T} : \Omega \subset \mathcal{H} \rightarrow \mathcal{H}$  is assumed to be linear and injective. It is well known by the theory of inverse problems that a regularization strategy is admissible if the regularization parameter satisfies,

$$\alpha(\delta, f^\delta) = \sup\{\alpha > 0 \mid \|\mathcal{T}\varphi_{\alpha(\delta)} - f^\delta\| \leq \tau\delta\}, \quad (1.3)$$

where  $\tau \geq 1$ , [16, Eq. (4.57) and (4.58)], [24, Definition 2.3]. The regularized solution  $\varphi_{\alpha(\delta)}$  of the problem (1.1) must satisfy the following first order optimality conditions,

$$\begin{aligned} 0 &= \nabla F_\alpha(\varphi_{\alpha(\delta)}) \\ 0 &= \mathcal{T}^*(\mathcal{T}\varphi_{\alpha(\delta)} - f^\delta) + \alpha(\delta) \nabla J(\varphi_{\alpha(\delta)}) \\ \mathcal{T}^*(f^\delta - \mathcal{T}\varphi_{\alpha(\delta)}) &= \alpha(\delta) \nabla J(\varphi_{\alpha(\delta)}). \end{aligned} \quad (1.4)$$

This work aims to answer two fundamental questions in the field of regularization theory; Is it possible to quantify  $\tau$  in (1.3) when the penalizer is (1.2)? What is the rule for the choice of regularization parameter  $\alpha(\delta, f^\delta)$  when the penalizer is (1.2) that the smoothed-TV is also an admissible regularization theory? We will be able to quantify the rate of the convergence of  $\|\varphi_{\alpha(\delta)} - \varphi^\dagger\|$  by means of the Bregman divergence.

Existence of the solution to the TV minimization problem, *i.e.*  $J(\cdot) = \int_{\Omega} \|\nabla(\cdot)\|_2 dx$  in the problem (1.1), has been discussed extensively [22, 29]. Moreover, an existence and uniqueness theorem for the minimizer of quadratic functionals with different type of convex integrands has been established in [11, Theorem 9.5-2]. As has been given by the *Minimal Hypersurfaces* problem in [13], the minimizer of the problem (1.1), for the smoothed-TV penalty  $J(\cdot) = \int_{\Omega} \sqrt{\|\nabla(\cdot)\|_2^2 + \beta} dx$ , exists on a reflexive Banach space.

## 2. Notations and Prerequisite Knowledge

### 2.1. Vector calculus notations

We assume to be tasked with reconstruction of some non-negative scalar function defined on a compact subset  $\Omega$  of  $\mathbb{R}^3$ , *i.e.*  $\varphi(\mathbf{x}) : \Omega \subset \mathbb{R}^3 \rightarrow \mathbb{R}_+$  where the spatial coordinate is  $\mathbf{x} = (x, y, z)$ . Then the gradient of  $\varphi$  is regarded as a vector with components

$$\nabla\varphi = \left( \frac{\partial\varphi}{\partial x}, \frac{\partial\varphi}{\partial y}, \frac{\partial\varphi}{\partial z} \right)^T.$$

The magnitude of this gradient in the Euclidean sense,

$$\|\nabla\varphi\|_2 = \left( \left| \frac{\partial\varphi}{\partial x} \right|^2 + \left| \frac{\partial\varphi}{\partial y} \right|^2 + \left| \frac{\partial\varphi}{\partial z} \right|^2 \right)^{1/2}. \quad (2.1)$$

### 2.2. Functional analysis notations

We aim to approximate a function which belongs to Hölder space. Hölder space is denoted by  $\mathcal{C}^{0,\gamma}(\Omega)$  where  $0 < \gamma \leq 1$ , [17, Subsection 5.1]. If a multivariate function  $\varphi(\mathbf{x}) \in \mathcal{C}^{0,\gamma}(\Omega)$ , then there exists  $\kappa > 0$  such that the function  $\varphi(\mathbf{x})$  satisfies the following Hölder continuity

$$|\varphi(\mathbf{x}) - \varphi(\tilde{\mathbf{x}})| \leq \kappa \|\mathbf{x} - \tilde{\mathbf{x}}\|_2^\gamma, \quad \forall \mathbf{x}, \tilde{\mathbf{x}} \in \Omega. \quad (2.2)$$

Here  $|\cdot|$  is the absolute value of  $\varphi(\mathbf{x}) : \Omega \subset \mathbb{R}^3 \rightarrow \mathbb{R}_+$ . Hölder space is a Banach space endowed with the norm

$$\|\varphi\|_\gamma := \|\varphi\|_\infty + [\varphi]_{\mathcal{C}^{0,\gamma}}, \quad (2.3)$$

where the Hölder coefficient  $[\varphi]_{\mathcal{C}^{0,\gamma}(\Omega)}$  is defined by

$$[\varphi]_{\mathcal{C}^{0,\gamma}(\Omega)} := \sup_{\mathbf{x}, \tilde{\mathbf{x}} \in \Omega \subset \mathbb{R}^3} \frac{|\varphi(\mathbf{x}) - \varphi(\tilde{\mathbf{x}})|}{\|\mathbf{x} - \tilde{\mathbf{x}}\|_2^\gamma}, \quad (2.4)$$

and the Euclidean norm is

$$\|\mathbf{x} - \tilde{\mathbf{x}}\|_2^\gamma := ((x - \tilde{x}_o)^2 + (y - \tilde{y}_o)^2 + (z - \tilde{z}_o)^2)^{\gamma/2}. \quad (2.5)$$

So that, we define Hölder space by

$$\mathcal{C}^{0,\gamma}(\Omega) := \{\varphi \in \mathcal{C}(\Omega) : \|\varphi\|_\gamma < \infty\}.$$

In this work, we focus on total variation (TV) of a function, [8, 28]. With (2.1), TV of our multivariate function is explicitly,

$$TV(\varphi) := \int_{\Omega} \|\nabla\varphi\|_2 dx = \int_{\Omega_x} \int_{\Omega_y} \int_{\Omega_z} \left( \left( \frac{\partial\varphi}{\partial x} \right)^2 + \left( \frac{\partial\varphi}{\partial y} \right)^2 + \left( \frac{\partial\varphi}{\partial z} \right)^2 \right)^{1/2} dx dy dz.$$

Total variation type regularization targets the reconstruction of bounded variation (BV) class of functions, [30],

$$\|\varphi\|_{BV} := \|\varphi\|_{\mathcal{L}^1} + TV(\varphi). \quad (2.6)$$

### 2.3. Bregman divergence

Following formulation emphasizes the functionality of the Bregman divergence in proving the norm convergence of the minimizer of the convex minimization problem to the true solution.

DEFINITION 2.1 (Total convexity and Bregman divergence). [6, Definition 1]

Let  $\Phi : \mathcal{H} \rightarrow \mathbb{R} \cup \{\infty\}$  be a smooth and convex functional. Then  $\Phi$  is called totally convex in  $u^* \in \mathcal{H}$ , if, for  $\nabla\Phi(u^*)$  and  $\{u\}$ , it holds that

$$D_\Phi(u, u^*) = \Phi(u) - \Phi(u^*) - \langle \nabla\Phi(u^*), u - u^* \rangle \rightarrow 0 \Rightarrow \|u - u^*\|_{\mathcal{H}} \rightarrow 0$$

where  $D_\Phi(u, u^*)$  represents the Bregman divergence.

It is said that  $\Phi$  is  $q$ -convex in  $u^* \in \mathcal{H}$  with a  $q \in [2, \infty)$ , if for all  $M > 0$  there exists a  $c^* > 0$  such that for all  $\|u - u^*\|_{\mathcal{H}} \leq M$  we have

$$D_\Phi(u, u^*) = \Phi(u) - \Phi(u^*) - \langle \nabla\Phi(u^*), u - u^* \rangle \geq c^* \|u - u^*\|_{\mathcal{H}}^q. \quad (2.7)$$

Throughout our norm convergence estimations, we refer to this definition for the case of 2-convexity.

In fact, another similar estimation to (2.7), for  $q = 2$ , can also be derived by making further assumption about the functional  $\Phi$  one of which is strong convexity with modulus  $c$ , [5, Definition 10.5]. Below is this alternative way of obtaining (2.7) when  $q = 2$ .

PROPOSITION 2.2. Let  $\Phi : \mathcal{H} \rightarrow \mathbb{R} \cup \{\infty\}$  be  $\Phi \in \mathcal{C}^2(\mathcal{H})$  is strongly convex with modulus of convexity  $c > 0$ , i.e.  $\nabla^2\Phi \succ cI$ , then

$$D_\Phi(u, v) > c\|u - v\|^2 + \mathcal{O}(\|u - v\|^2). \quad (2.8)$$

*Proof.* Let us begin with considering the Taylor expansion of  $\Phi$ ,

$$\Phi(u) = \Phi(v) + \langle \nabla\Phi(v), u - v \rangle + \frac{1}{2} \langle \nabla^2\Phi(v)(u - v), u - v \rangle + \mathcal{O}(\|u - v\|^2). \quad (2.9)$$

Then the Bregman divergence

$$\begin{aligned} D_\Phi(u, v) &= \Phi(u) - \Phi(v) - \langle \nabla\Phi(v), u - v \rangle \\ &= \langle \nabla\Phi(v), u - v \rangle + \frac{1}{2} \langle \nabla^2\Phi(v)(u - v), u - v \rangle + \mathcal{O}(\|u - v\|^2) - \langle \nabla\Phi(v), u - v \rangle \\ &= \frac{1}{2} \langle \nabla^2\Phi(v)(u - v), u - v \rangle + \mathcal{O}(\|u - v\|^2). \end{aligned}$$

Since  $\Phi(\cdot)$  is strictly convex, due to strong convexity and  $\Phi \in \mathcal{C}^2(\mathcal{H})$ , hence one obtains that

$$D_\Phi(u, v) > c\|u - v\|^2 + \mathcal{O}(\|u - v\|^2), \quad (2.10)$$

where  $c$  is the modulus of convexity.

□

#### 2.4. Further Results on the Hölder Continuity

We already have reviewed in Subsection 2.2 that the Hölder space  $\mathcal{C}^{0,\gamma}$  is a Banach space endowed with the norm, for all  $x \neq y \in \Omega$  and  $\Omega$  is a compact domain,

$$\|\varphi\|_\gamma := \sup_{\mathbf{x} \in \Omega} |\varphi(\mathbf{x})| + [\varphi]_{\mathcal{C}^{0,\gamma}(\Omega)} = \|\varphi\|_\infty + [\varphi]_{\mathcal{C}^{0,\gamma}(\Omega)}. \quad (2.11)$$

Here the Hölder coefficient is obviously bounded by

$$[\varphi]_{\mathcal{C}^{0,\gamma}(\Omega)} := \sup_{\mathbf{x}, \tilde{\mathbf{x}} \in \Omega} \frac{|\varphi(\mathbf{x}) - \varphi(\tilde{\mathbf{x}})|}{\|\mathbf{x} - \tilde{\mathbf{x}}\|_2^\gamma} \leq \kappa. \quad (2.12)$$

Furthermore, following from (2.11), an immediate conclusion can be formulated as follows.

**PROPOSITION 2.3.** *Over the compact domain  $\Omega$ , if  $\varphi \in \mathcal{C}^{0,\gamma}(\Omega)$ , then  $\varphi \in \mathcal{L}^1(\Omega)$ .*

*Proof.* Since  $\|\varphi\|_\infty \geq \frac{1}{|\Omega|} \|\varphi\|_{\mathcal{L}^1}$  and  $[\varphi]_{\mathcal{C}^{0,\gamma}(\Omega)} > 0$ , then

$$\|\varphi\|_\gamma \geq \|\varphi\|_\infty \geq \frac{1}{|\Omega|} \|\varphi\|_{\mathcal{L}^1}. \quad (2.13)$$

□

### 3. Hölder Continuity and TV of a $\mathcal{C}^1$ –Smooth Function

We now come to the point where we start establishing the relations between  $\gamma$ –Hölder continuity and TV of a function  $\varphi$  on  $\mathbb{R}^3$ . The following theorems will also serve us for determining an implementable and unique regularization parameter appeared in the minimization problem (1.1). We emphasize a very important assumption that we always work with continuous function on a compact domain which is uniformly continuous. This fact will allow us to interchange the necessary operations in order to obtain the desired results in what follows.

**THEOREM 3.1** (Morrey’s inequality). *[17, Subsection 5.6.2., Theorem 4] Let  $\Omega \subset \mathbb{R}^N$  be the compact domain and let  $N < p \leq \infty$ . Then there exists a constant  $C$ , depending only on  $p$  and  $N$ , such that*

$$\|\varphi\|_{\mathcal{C}^{0,\gamma}(\Omega)} \leq C \|\varphi\|_{\mathcal{W}^{1,p}(\Omega)} \quad (3.1)$$

for all  $\varphi \in \mathcal{C}^1(\Omega)$ , where

$$\gamma := 1 - N/p. \quad (3.2)$$

**COROLLARY 3.2.** *Specifically in  $\mathbb{R}^3$ , the theorem implies that*

$$[\varphi]_{\mathcal{C}^{0,1/4}(\Omega)} \leq C \|\nabla \varphi\|_{\mathcal{L}^4(\Omega)}, \quad (3.3)$$

since  $\gamma := 1 - 3/4$ .

**THEOREM 3.3.** *Over the compact domain  $\Omega \subset \mathbb{R}^3$ , with its volume  $|\Omega| \in \mathbb{R}_+$ , let  $\varphi \in \mathcal{C}^1(\Omega) \cap \mathcal{C}^{0,1/4}(\Omega)$ . Then Hölder coefficient  $[\varphi]_{\mathcal{C}^{0,1/4}(\Omega)}$  of the function is bounded by its total variation  $TV(\varphi)$  as such,*

$$[\varphi]_{\mathcal{C}^{0,1/4}(\Omega)} \leq \frac{r(\Omega)^{3/4}}{|\Omega|} TV(\varphi), \text{ where } r(\Omega) := \sup_{\mathbf{x}, \tilde{\mathbf{x}} \in \Omega} \|\mathbf{x} - \tilde{\mathbf{x}}\|.$$

*Proof.* Recall our vectoral notations in  $\mathbb{R}^3$ ,  $\mathbf{x} = (x, y, z)$  and  $\tilde{\mathbf{x}} = (\tilde{x}, \tilde{y}, \tilde{z})$ . Then for a fixed  $\gamma \in (0, 1]$ , componentwise Hölder continuity in  $\mathbb{R}^3$  is given by

$$[\varphi]_{\mathcal{C}^{0,\gamma}(\Omega)} = \sup_{\mathbf{x}, \tilde{\mathbf{x}} \in \Omega \subset \mathbb{R}^3} \left\{ \frac{|\varphi(x, y, z) - \varphi(\tilde{x}_o, \tilde{y}, \tilde{z})|}{\|\mathbf{x} - \tilde{\mathbf{x}}\|_2^\gamma}, \frac{|\varphi(x, y, z) - \varphi(\tilde{x}, \tilde{y}_o, \tilde{z})|}{\|\mathbf{x} - \tilde{\mathbf{x}}\|_2^\gamma}, \dots \right. \\ \left. \dots, \frac{|\varphi(x, y, z) - \varphi(\tilde{x}, \tilde{y}, \tilde{z}_o)|}{\|\mathbf{x} - \tilde{\mathbf{x}}\|_2^\gamma} \right\}.$$

By the definition of Euclidean norm in (2.5),

$$\|\mathbf{x} - \tilde{\mathbf{x}}\|_2^\gamma \geq \sup_{\Omega} \{|x - \tilde{x}_o|^\gamma, |y - \tilde{y}_o|^\gamma, |z - \tilde{z}_o|^\gamma\}.$$

So this implies

$$[\varphi]_{\mathcal{C}^{0,\gamma}(\Omega)} \leq \sup_{\mathbf{x}, \tilde{\mathbf{x}} \in \Omega \subset \mathbb{R}^3} \left\{ \frac{|\varphi(x, y, z) - \varphi(\tilde{x}_o, \tilde{y}, \tilde{z})|}{|x - \tilde{x}_o|^\gamma}, \frac{|\varphi(x, y, z) - \varphi(\tilde{x}, \tilde{y}_o, \tilde{z})|}{|y - \tilde{y}_o|^\gamma}, \dots \right. \\ \left. \dots, \frac{|\varphi(x, y, z) - \varphi(\tilde{x}, \tilde{y}, \tilde{z}_o)|}{|z - \tilde{z}_o|^\gamma} \right\}, \\ = \sup_{\mathbf{x}, \tilde{\mathbf{x}} \in \Omega \subset \mathbb{R}^3} \left\{ \frac{|\varphi(x, y, z) - \varphi(\tilde{x}_o, \tilde{y}, \tilde{z})|}{|x - \tilde{x}_o||x - \tilde{x}_o|^{\gamma-1}}, \frac{|\varphi(x, y, z) - \varphi(\tilde{x}, \tilde{y}_o, \tilde{z})|}{|y - \tilde{y}_o||y - \tilde{y}_o|^{\gamma-1}}, \dots \right. \\ \left. \dots, \frac{|\varphi(x, y, z) - \varphi(\tilde{x}, \tilde{y}, \tilde{z}_o)|}{|z - \tilde{z}_o||z - \tilde{z}_o|^{\gamma-1}} \right\}, \\ = \sup_{\mathbf{x}, \tilde{\mathbf{x}} \in \Omega \subset \mathbb{R}^3} \left\{ \frac{|\varphi(x, y, z) - \varphi(\tilde{x}_o, \tilde{y}, \tilde{z})|}{|x - \tilde{x}_o|} |x - \tilde{x}_o|^{1-\gamma}, \frac{|\varphi(x, y, z) - \varphi(\tilde{x}, \tilde{y}_o, \tilde{z})|}{|y - \tilde{y}_o|} |y - \tilde{y}_o|^{1-\gamma}, \dots \right. \\ \left. \dots, \frac{|\varphi(x, y, z) - \varphi(\tilde{x}, \tilde{y}, \tilde{z}_o)|}{|z - \tilde{z}_o|} |z - \tilde{z}_o|^{1-\gamma} \right\}.$$

Here, the last equality in the chain is rather convenient to present since  $\gamma - 1 < 0 < 1$ . Obviously, for any pair of points  $(\mathbf{x}, \tilde{\mathbf{x}}) \in \Omega$ , there exists  $s > 0$  such that  $s = \|\mathbf{x} - \tilde{\mathbf{x}}\|_2$ . Then,

$$r^{1-\gamma} \geq s^{1-\gamma} = \|\mathbf{x} - \tilde{\mathbf{x}}\|_2^{1-\gamma} \geq \sup_{\Omega} \{|x - \tilde{x}_o|^{1-\gamma}, |y - \tilde{y}_o|^{1-\gamma}, |z - \tilde{z}_o|^{1-\gamma}\},$$

we have

$$\begin{aligned}
[\varphi]_{C^{0,\gamma}(\Omega)} &\leq \sup_{\mathbf{x}, \tilde{\mathbf{x}} \in \Omega \subset \mathbb{R}^3} \left\{ \frac{|\varphi(x, y, z) - \varphi(\tilde{x}_o, \tilde{y}, \tilde{z})|}{|x - \tilde{x}_o|} \|\mathbf{x} - \tilde{\mathbf{x}}\|_2^{1-\gamma} \right. \\
&\quad, \frac{|\varphi(x, y, z) - \varphi(\tilde{x}, \tilde{y}_o, \tilde{z})|}{|y - \tilde{y}_o|} \|\mathbf{x} - \tilde{\mathbf{x}}\|_2^{1-\gamma} \\
&\quad, \left. \frac{|\varphi(x, y, z) - \varphi(\tilde{x}, \tilde{y}, \tilde{z}_o)|}{|z - \tilde{z}_o|} \|\mathbf{x} - \tilde{\mathbf{x}}\|_2^{1-\gamma} \right\} \\
&= \sup_{\mathbf{x}, \tilde{\mathbf{x}} \in \Omega \subset \mathbb{R}^3} \left\{ \frac{|\varphi(x, y, z) - \varphi(\tilde{x}_o, \tilde{y}, \tilde{z})|}{|x - \tilde{x}_o|} s^{1-\gamma}, \frac{|\varphi(x, y, z) - \varphi(\tilde{x}, \tilde{y}_o, \tilde{z})|}{|y - \tilde{y}_o|} s^{1-\gamma}, \right. \\
&\quad, \left. \frac{|\varphi(x, y, z) - \varphi(\tilde{x}, \tilde{y}, \tilde{z}_o)|}{|z - \tilde{z}_o|} s^{1-\gamma} \right\}, \\
&\leq r^{1-\gamma} \sup_{\mathbf{x}, \tilde{\mathbf{x}} \in \Omega \subset \mathbb{R}^3} \left\{ \frac{|\varphi(x, y, z) - \varphi(\tilde{x}_o, \tilde{y}, \tilde{z})|}{|x - \tilde{x}_o|}, \frac{|\varphi(x, y, z) - \varphi(\tilde{x}, \tilde{y}_o, \tilde{z})|}{|y - \tilde{y}_o|}, \right. \\
&\quad, \left. \frac{|\varphi(x, y, z) - \varphi(\tilde{x}, \tilde{y}, \tilde{z}_o)|}{|z - \tilde{z}_o|} \right\}.
\end{aligned}$$

Recall that our function  $\varphi$  is continuous over the compact domain  $\Omega$  which makes it uniformly continuous on the same domain. Then we are allowed to interchange  $\lim$  with  $\sup$ . Now, moving on to the limit on both sides with respect to each component  $\lim_{x \rightarrow \tilde{x}_o}$ ,  $\lim_{y \rightarrow \tilde{y}_o}$  and  $\lim_{z \rightarrow \tilde{z}_o}$ ,

$$\begin{aligned}
[\varphi]_{C^{0,\gamma}(\Omega)} &\leq r^{1-\gamma} \left( \left| \frac{\partial \varphi}{\partial x} \right|, \left| \frac{\partial \varphi}{\partial y} \right|, \left| \frac{\partial \varphi}{\partial z} \right| \right) \\
&= r^{1-\gamma} \left( \left( \left| \frac{\partial \varphi}{\partial x} \right|^2 \right)^{1/2}, \left( \left| \frac{\partial \varphi}{\partial y} \right|^2 \right)^{1/2}, \left( \left| \frac{\partial \varphi}{\partial z} \right|^2 \right)^{1/2} \right) \\
&\leq r^{1-\gamma} \|\nabla \varphi\|_2
\end{aligned}$$

Again, the last inequality has been obtained by the fact that sum of the components always remains greater than each component itself. Now, integrate both sides over the compact domain  $\Omega$  to yield

$$[\varphi]_{C^{0,\gamma}(\Omega)} \leq \frac{r^{1-\gamma}}{|\Omega|} TV(\varphi),$$

which is, to be more precise,

$$[\varphi]_{C^{0,1/4}(\Omega)} \leq \frac{r^{3/4}}{|\Omega|} TV(\varphi),$$

since  $\gamma = 1/4$  in  $\mathbb{R}^3$ .

□

This shows that Hölder coefficient of a function  $\varphi \in \mathcal{C}^1(\Omega) \cap \mathcal{C}^{0,1/4}(\Omega)$  is an approximation for the total variation of the same function. In the following theorems, we will establish the reverse direction of this statement. To do so, we will make use of the Lipschitz continuity which is a specific case of Hölder continuity in (2.2) for  $\gamma = 1$ .

**THEOREM 3.4.** *Under the same conditions of Theorem 3.3, for  $\gamma = 1$  in (2.2),*

$$\|\nabla\varphi\|_{\mathcal{L}^1} \leq \kappa|\Omega|. \quad (3.4)$$

*Proof.* As we have introduced in the Section 2 by (2.1),

$$\|\nabla\varphi\|_2 = \left( \left| \frac{\partial\varphi}{\partial x} \right|^2 + \left| \frac{\partial\varphi}{\partial y} \right|^2 + \left| \frac{\partial\varphi}{\partial z} \right|^2 \right)^{1/2} \leq \left| \frac{\partial\varphi}{\partial x} \right| + \left| \frac{\partial\varphi}{\partial y} \right| + \left| \frac{\partial\varphi}{\partial z} \right|. \quad (3.5)$$

This inequality has been obtained by using the following simple identity

$$(p + q + s)^2 = p^2 + q^2 + s^2 + 2pq + 2s(p + q) \geq p^2 + q^2 + s^2,$$

for  $p, q, s \in \mathbb{R}_+$ . This implies

$$p + q + s \geq (p^2 + q^2 + s^2)^{1/2}.$$

To arrive at (3.5), set  $p := \left| \frac{\partial\varphi}{\partial x} \right|$ ,  $q := \left| \frac{\partial\varphi}{\partial y} \right|$ , and lastly  $s := \left| \frac{\partial\varphi}{\partial z} \right|$ . Now by the definition of partial derivative in the componentwise sense,

$$\begin{aligned} \|\nabla\varphi\|_2 &\leq \lim_{x \rightarrow \tilde{x}_o} \frac{|\varphi(x, y, z) - \varphi(\tilde{x}_o, \tilde{y}, \tilde{z})|}{|x - \tilde{x}_o|} + \lim_{y \rightarrow \tilde{y}_o} \frac{|\varphi(x, y, z) - \varphi(\tilde{x}, \tilde{y}_o, \tilde{z})|}{|y - \tilde{y}_o|} + \\ &\quad + \lim_{z \rightarrow \tilde{z}_o} \frac{|\varphi(x, y, z) - \varphi(\tilde{x}, \tilde{y}, \tilde{z}_o)|}{|z - \tilde{z}_o|} \end{aligned}$$

Gradient of the functional  $\varphi(\mathbf{x})$  over the compact domain is valid for any  $\tilde{\mathbf{x}}_o \in \Omega$ . Therefore, we continue with our proof in the unified form. First observe that by the Lebesgue dominated convergence theorem,

$$\begin{aligned} \int_{\Omega} \|\nabla\varphi(\tilde{\mathbf{x}}_o)\|_2 d\tilde{\mathbf{x}}_o &\leq \int_{\Omega} \lim_{\mathbf{x} \rightarrow \tilde{\mathbf{x}}_o} \left\{ \frac{|\varphi(\mathbf{x}) - \varphi(\tilde{\mathbf{x}}_o)|}{\|\mathbf{x} - \tilde{\mathbf{x}}_o\|_2} \right\} d\tilde{\mathbf{x}}_o \\ &= \lim_{\mathbf{x} \rightarrow \tilde{\mathbf{x}}_o} \int_{\Omega} \left\{ \frac{|\varphi(\mathbf{x}) - \varphi(\tilde{\mathbf{x}}_o)|}{\|\mathbf{x} - \tilde{\mathbf{x}}_o\|_2} \right\} d\tilde{\mathbf{x}}_o. \end{aligned} \quad (3.6)$$

Since  $\varphi \in \mathcal{C}^1(\Omega)$ , then Hölder continuity given by (2.2) is satisfied for  $\gamma = 1$ ,

$$|\varphi(\mathbf{x}) - \varphi(\tilde{\mathbf{x}}_o)| \leq \kappa \|\mathbf{x} - \tilde{\mathbf{x}}_o\|_2$$

which is Lipschitz continuity. Then (3.6) reads

$$\int_{\Omega} \|\nabla\varphi(\tilde{\mathbf{x}}_o)\|_2 d\tilde{\mathbf{x}}_o \leq \lim_{\mathbf{x} \rightarrow \tilde{\mathbf{x}}_o} \int_{\Omega} \kappa d\tilde{\mathbf{x}}_o = \kappa|\Omega|. \quad (3.7)$$

□



We formulate the last formulatin for this section which is an immediate consequence of this theorem.

COROLLARY 3.5. *Under the same conditions of Theorem 3.4, then,*

$$\int_{\Omega} \|\nabla \varphi\|_2^2 dx \leq \kappa^2 |\Omega|^2. \quad (3.8)$$

*Proof.* Again, by the definition of Euclidean norm in Section 2 by (2.1),

$$\|\nabla \varphi\|_2^2 = \left| \frac{\partial \varphi}{\partial x} \right|^2 + \left| \frac{\partial \varphi}{\partial y} \right|^2 + \left| \frac{\partial \varphi}{\partial z} \right|^2.$$

Analogous to the proof of Theorem 3.4,

$$\begin{aligned} \int_{\Omega} \|\nabla \varphi(\tilde{\mathbf{x}}_o)\|_2^2 d\tilde{\mathbf{x}}_o &\leq \int_{\Omega} \lim_{\mathbf{x} \rightarrow \tilde{\mathbf{x}}_o} \left\{ \frac{|\varphi(\mathbf{x}) - \varphi(\tilde{\mathbf{x}}_o)|}{\|\mathbf{x} - \tilde{\mathbf{x}}_o\|_2} \right\}^2 d\tilde{\mathbf{x}}_o \\ &= \lim_{\mathbf{x} \rightarrow \tilde{\mathbf{x}}_o} \int_{\Omega} \left\{ \frac{|\varphi(\mathbf{x}) - \varphi(\tilde{\mathbf{x}}_o)|}{\|\mathbf{x} - \tilde{\mathbf{x}}_o\|_2} \right\}^2 d\tilde{\mathbf{x}}_o \\ &\leq \kappa^2 |\Omega|^2 \end{aligned} \quad (3.9)$$

since  $\varphi(\mathbf{x}) \in \mathcal{C}^1(\Omega)$ . □

#### 4. Smoothed-TV Regularization Is an Admissible Regularization Strategy With the Hölder Continuity

We will define such a regularization parameter which will simultaneously enable us to prove the convergence of the smoothed-TV regularization and to estimate the discrepancy  $\|\mathcal{T}\varphi - f^\delta\|$  for the corresponding regularization strategy, [10]. Unlike the available literature, [1, 3, 4, 8, 9, 10, 14, 15, 31], we define discrepancy principle for the smoothed-TV regularization under a particular rule for the choice of regularization parameter. Furthermore, still with the same regularization parameter, we manage to show that smoothed-TV regularization is an admissible regularization strategy with Hölder continuity. Throughout this section, the fact that our targeted solution function is Hölder continuous will be to our benefit to be able provide an implementable regularization parameter for copmputerized environment. Hereafter, the component  $\mathbf{x}$  is replaced by  $x$  only for the sake of simplicity.

To be able to show the convergence of  $\|\varphi_{\alpha(\delta)} - \varphi^\dagger\|$ , we will refer to Bregman divergence. In Proposition 2.2, we have demonstrated the relation between strong convexity and 2-convexity. Convexity of the smoothed total variation penalizer has been established in [1, Theorem 2.4]. We will ensure the strong convexity of the same penalizer in the following formulation.

THEOREM 4.1. *For any  $\beta > 0$ , the functional  $J(\varphi) := \int_{\Omega} \sqrt{\|\nabla \varphi\|_2^2 + \beta} dx$  is strongly convex.*

*Proof.* It suffices to prove that  $\nabla^2 J(\varphi) \geq 0$ . To avoid confusion in the calculations, we will make an assignment  $g(p) = \sqrt{|p|^2 + \beta}$  where  $p = \nabla \varphi$ . According to Leibniz integral rule, calculating  $\nabla^2 J(\varphi) \geq 0$  and  $g''(p)$  are equivalent to each other. Then

$$g'(p) = \frac{p}{\sqrt{p^2 + \beta}},$$

and likewise

$$g''(p) = \frac{\beta}{(p^2 + \beta)^{3/2}}.$$

Obviously,  $g''(p) > 0$  for any  $\beta > 0$ .

□

**THEOREM 4.2.** *Over the compact domain  $\Omega \subset \mathbb{R}^3$ , assume that  $u, v \in \mathcal{C}^1(\Omega) \cap \mathcal{C}^{0,\gamma}(\Omega)$ . Then there exists a dynamical positive real-valued functional  $\mathcal{K}(u) : \mathcal{C}^1(\Omega) \rightarrow \mathbb{R}_+$  depending on  $u$  such that*

$$J(u) - J(v) \leq 2\kappa^2 |\Omega|^2 \mathcal{K}(u) \|\nabla(u - v)\|_2,$$

for  $J(\cdot)$  defined by  $J(\cdot) = \int_{\Omega} \sqrt{\|\nabla(\cdot)\|_2^2 + \beta} dx$  and where  $\kappa$  satisfies (2.2) for  $\gamma = 1$ .

*Proof.* By the definition of  $J(\cdot) := \int_{\Omega} \sqrt{\|\nabla \cdot\|_2^2 + \beta} dx$ ,

$$\begin{aligned} J(u) - J(v) &= \int_{\Omega} \frac{\|\nabla u\|_2^2 - \|\nabla v\|_2^2}{(\|\nabla u\|_2^2 + \beta)^{1/2} + (\|\nabla v\|_2^2 + \beta)^{1/2}} dx \\ &\leq \int_{\Omega} \frac{\|\nabla u\|_2^2 - \|\nabla v\|_2^2}{(\|\nabla v\|_2^2 + \beta)^{1/2}} dx. \end{aligned} \tag{4.1}$$

Now choose  $\mathcal{U} = \min_{x \in \Omega} \{\|\nabla u\|_2^2\}$  to have,

$$\begin{aligned} J(u) - J(v) &\leq \frac{1}{(\mathcal{U} + \beta)^{1/2}} \int_{\Omega} (\|\nabla u\|_2^2 - \|\nabla v\|_2^2) dx \\ &= \frac{1}{(\mathcal{U} + \beta)^{1/2}} \int_{\Omega} (\|\nabla u\|_2 - \|\nabla v\|_2)(\|\nabla u\|_2 + \|\nabla v\|_2) dx \\ &\leq \frac{1}{(\mathcal{U} + \beta)^{1/2}} \int_{\Omega} (\|\nabla u - \nabla v\|_2)(\|\nabla u\|_2 + \|\nabla v\|_2) dx. \end{aligned}$$

Apply Hölder inequality to have,

$$\begin{aligned} J(u) - J(v) &\leq \frac{1}{(\mathcal{U} + \beta)^{1/2}} \left( \int_{\Omega} \|\nabla u - \nabla v\|_2^2 dx \right)^{1/2} \left( \int_{\Omega} (\|\nabla u\|_2 + \|\nabla v\|_2)^2 dx \right)^{1/2} \\ &\leq \frac{2}{(\mathcal{U} + \beta)^{1/2}} \left( \int_{\Omega} \|\nabla u - \nabla v\|_2^2 dx \right)^{1/2} \left( \int_{\Omega} \|\nabla u\|_2^2 + \|\nabla v\|_2^2 dx \right)^{1/2}, \end{aligned}$$

since  $2ab \leq a^2 + b^2$  for any  $a, b \in \mathbb{R}_+$ . By Corollary 3.5, we have already obtained the upper bound for the second integral on the right hand side. Then,

$$\begin{aligned} J(u) - J(v) &\leq \frac{2\kappa^2|\Omega|^2}{(\mathcal{U} + \beta)^{1/2}} \left( \int_{\Omega} \|\nabla u - \nabla v\|_2^2 dx \right)^{1/2} \\ &= \frac{2\kappa^2|\Omega|^2}{(\mathcal{U} + \beta)^{1/2}} \|\nabla(u - v)\|_2. \end{aligned}$$

Hence, the positive real valued functional is defined by,

$$\mathcal{K}(u) := \frac{1}{(\mathcal{U} + \beta)^{1/2}}.$$

□

An immediate consequence that we make use of  $\mathcal{C}^{1+}(\Omega)$  function space is formulated below.

**COROLLARY 4.3.** *Over the compact domain  $\Omega \subset \mathbb{R}^3$ , assume that  $u, v \in \mathcal{C}^{1+}(\Omega) \cap \mathcal{C}^{0,\gamma}(\Omega)$ . Then with the same functional  $\mathcal{K}(u) : \mathcal{C}^{1+}(\Omega) \rightarrow \mathbb{R}_+$  as appears in Theorem 4.2, it is hold that*

$$J(u) - J(v) \leq 2\kappa^2 L_{u,v} |\Omega|^2 \mathcal{K}(u),$$

for  $J(\cdot)$  defined by  $J(\cdot) = \int_{\Omega} \sqrt{\|\nabla(\cdot)\|_2^2 + \beta} dx$ , where  $\kappa$  satisfies (2.2) for  $\gamma = 1$ .

*Proof.* Since  $u, v \in \mathcal{C}^{1+}(\Omega)$ , there there exists constant satisfying  $\|\nabla(u - v)\|_2 \leq L_{u,v} \|u - v\|_2$ . Then it follows from the above calculations in the proof of Theorem 4.2,

$$\begin{aligned} J(u) - J(v) &\leq \frac{2\kappa^2|\Omega|^2}{(\mathcal{U} + \beta)^{1/2}} \|\nabla(u - v)\|_2 \\ &\leq \frac{2\kappa^2|\Omega|^2}{(\mathcal{U} + \beta)^{1/2}} L_{u,v} \|u - v\|_2. \end{aligned}$$

□

#### 4.1. Discrepancy principle for the smoothed TV regularizer

We are able to evaluate the fixed coefficient  $\tau$  in the discrepancy principle  $\|\mathcal{T}\varphi_{\alpha(\delta)} - f^\delta\| \leq \tau\delta$  for the smoothed TV penalty  $J(\cdot)$  in the problem (1.1). To do so, we need to assume that the target function to be class of  $\mathcal{C}^{1+}(\Omega)$ .

Moreover, in order for a precise upper bound for  $\|\mathcal{T}\varphi_{\alpha(\delta)} - f^\delta\|$ , we will need to focus on our specified penalty  $J(\varphi) := \int_{\Omega} \sqrt{\|\nabla\varphi\|_2^2 + \beta} dx$ . The regularized solution  $\varphi_{\alpha(\delta)}$  to the problem (1.1) is the minimum of  $F(\varphi)$  for all  $\varphi \in \mathcal{D}(\mathcal{T})$ . Which is in other words,

$$\varphi_{\alpha(\delta)} \in \arg \min_{\varphi} \left\{ F_{\alpha}(\varphi) = \frac{1}{2} \|\mathcal{T}\varphi - f^{\delta}\|_{\mathcal{L}^2}^2 + \alpha J(\varphi) \right\}.$$

Then

$$\frac{1}{2} \|\mathcal{T}\varphi_{\alpha(\delta)} - f^{\delta}\|_{\mathcal{L}^2}^2 + \alpha J(\varphi_{\alpha(\delta)}) \leq \frac{1}{2} \|\mathcal{T}\varphi^{\dagger} - f^{\delta}\|_{\mathcal{L}^2}^2 + \alpha J(\varphi^{\dagger}). \quad (4.2)$$

Since the true data  $f^{\dagger}$  satisfying the operator equation  $\mathcal{T}\varphi^{\dagger} = f^{\dagger}$  lies in some  $\delta$ -ball  $\mathcal{B}_{\delta}(f^{\delta})$ , i.e.  $\|f^{\dagger} - f^{\delta}\| \leq \delta$ , then (4.2) reads,

$$\frac{1}{2} \|\mathcal{T}\varphi_{\alpha(\delta)} - f^{\delta}\|_{\mathcal{L}^2}^2 \leq \frac{1}{2} \delta^2 + \alpha(J(\varphi^{\dagger}) - J(\varphi_{\alpha(\delta)})). \quad (4.3)$$

Further development of this estimation will be done by means of Theorem 3.4 as formulated below.

**THEOREM 4.4** (Discrepancy principle for the smoothed-TV regularization). *Over the compact domain  $\Omega \subset \mathbb{R}^3$ , denote by  $\varphi_{\alpha(\delta)}, \varphi^{\dagger} \in \mathcal{C}^{1+}(\Omega) \cap \mathcal{C}^{0,\gamma}(\Omega)$  the regularized and the true solutions to the problem (1.1) respectively. If the regularization parameter  $\alpha(\delta)$  is chosen according to the rule of,*

$$\alpha(\delta) \leq \delta^2 \frac{(\mathcal{K}(\varphi_{\alpha(\delta)}))^{-1}}{2\kappa^2 |\Omega|^2}, \quad (4.4)$$

for  $\kappa$  satisfying (2.2) with  $\gamma = 1$ , and

$$\mathcal{K}(\varphi_{\alpha(\delta)}) := \frac{1}{(\mathcal{U} + \beta)^{1/2}}$$

where

$$\mathcal{U} = \min_{x \in \Omega} \{ \|\nabla \varphi_{\alpha(\delta)}\|_2^2 \},$$

then the discrepancy  $\|\mathcal{T}\varphi_{\alpha(\delta)} - f^{\delta}\|_{\mathcal{L}^2}$  for the smoothed-TV regularization is estimated by,

$$\|\mathcal{T}\varphi_{\alpha(\delta)} - f^{\delta}\|_{\mathcal{L}^2} \leq \delta \sqrt{1 + \|\nabla(\varphi_{\alpha(\delta)} - \varphi^{\dagger})\|_2}. \quad (4.5)$$

Furthermore, if the regularization parameter fulfils

$$\alpha(\delta) \leq \delta^2 \frac{(\mathcal{K}(\varphi_{\alpha(\delta)}))^{-1}}{2\kappa^2 |\Omega|^2 L_{\varphi_{\alpha(\delta)}, \varphi^{\dagger}}}, \quad (4.6)$$

where  $L_{\varphi_{\alpha(\delta)}, \varphi^{\dagger}}$  is an appropriate Lipschitz constant then,

$$\|\mathcal{T}\varphi_{\alpha(\delta)} - f^{\delta}\|_{\mathcal{L}^2} \leq \delta \sqrt{1 + \|\varphi_{\alpha(\delta)} - \varphi^{\dagger}\|_2}. \quad (4.7)$$

*Proof.* From the calculations in (4.3) and the quick adaptation of Theorem 4.2, it is firstly obtained that

$$\begin{aligned} \|\mathcal{T}\varphi_{\alpha(\delta)} - f^\delta\|_{\mathcal{L}^2}^2 &\leq \delta^2 + 2\alpha(J(\varphi_{\alpha(\delta)}) - J(\varphi^\dagger)) \\ &\leq \delta^2 + 2\alpha\kappa^2|\Omega|^2\mathcal{K}(\varphi_{\alpha(\delta)})\|\nabla(\varphi_{\alpha(\delta)} - \varphi^\dagger)\|_2. \end{aligned} \quad (4.8)$$

Then, with the given rule of the regularization parameter  $\alpha$  in (4.4),

$$\|\mathcal{T}\varphi_{\alpha(\delta)} - f^\delta\|_{\mathcal{L}^2}^2 \leq \delta^2 + \delta^2\|\nabla(\varphi_{\alpha(\delta)} - \varphi^\dagger)\|_2,$$

which is the first result. It is not difficult to obtain the second part of the theorem. Analogous to Corollary 4.3, observe that there exists  $L_{\varphi_{\alpha(\delta)}, \varphi^\dagger}$  such that

$$\begin{aligned} J(\varphi_{\alpha(\delta)}) - J(\varphi^\dagger) &\leq \kappa^2|\Omega|^2\mathcal{K}(\varphi_{\alpha(\delta)})\|\nabla(\varphi_{\alpha(\delta)} - \varphi^\dagger)\|_2 \\ &\leq \kappa^2|\Omega|^2\mathcal{K}(\varphi_{\alpha(\delta)})L_{\varphi_{\alpha(\delta)}, \varphi^\dagger}\|\varphi_{\alpha(\delta)} - \varphi^\dagger\|_2 \end{aligned} \quad (4.9)$$

Then, from (4.8),

$$\|\mathcal{T}\varphi_{\alpha(\delta)} - f^\delta\|_{\mathcal{L}^2}^2 \leq \delta^2 + 2\alpha\kappa^2|\Omega|^2\mathcal{K}(\varphi_{\alpha(\delta)})L_{\varphi_{\alpha(\delta)}, \varphi^\dagger}\|\varphi_{\alpha(\delta)} - \varphi^\dagger\|_2. \quad (4.10)$$

Hence, with the given rule for choice of the regularization parameter  $\alpha$  in (4.6), the second desired result yields.  $\square$

Uniform continuity of the smoothed-TV regularization will come from formulating another useful Bregman divergence which will lead us to the ultimate result of this work. Before proceeding, it must be noted that the discrepancy principle for the smoothed TV is yet to be completed in (4.7) which is necessary in order for fulfilling the condition in (1.3). The completion will follow after quantifying the rate for  $\|\varphi_{\alpha(\delta)} - \varphi^\dagger\|_2$ .

**THEOREM 4.5.** *Under the assumptions of Theorem 4.4, if the regularization parameter, for  $\delta \in (0, 1)$ , satisfies*

$$\alpha(\delta) \leq \delta \frac{(\mathcal{K}(\varphi_{\alpha(\delta)}))^{-1}}{2\kappa^2|\Omega|^2L_{\varphi_{\alpha(\delta)}, \varphi^\dagger}},$$

*which is analogous to (4.6) in Theorem 4.4, then*

$$\|\varphi_{\alpha(\delta)} - \varphi^\dagger\|_2 \rightarrow 0, \text{ as } \alpha(\delta) \rightarrow 0 \text{ whilst } \delta \rightarrow 0.$$

*Proof.* Since we will prove the assertion depending on the choice of the regularization parameter, then we formulate another Bregman divergence associated with the functional  $\alpha J(\cdot)$ . Moreover, it is clear that first order optimality conditions in (1.4) must also hold for the true solution  $\varphi^\dagger$ . Then,

$$\begin{aligned} D_{\alpha J}(\varphi_{\alpha(\delta)}, \varphi^\dagger) &= \alpha J(\varphi_{\alpha(\delta)}) - \alpha J(\varphi^\dagger) - \langle \alpha \nabla J(\varphi^\dagger), \varphi_{\alpha(\delta)} - \varphi^\dagger \rangle \\ &= \alpha (J(\varphi_{\alpha(\delta)}) - J(\varphi^\dagger)) - \langle \mathcal{T}^*(f^\delta - \mathcal{T}\varphi^\dagger), \varphi_{\alpha(\delta)} - \varphi^\dagger \rangle \end{aligned} \quad (4.11)$$

Again, from the calculations in (4.3) and by Theorem 4.2, there exists Lipschitz constant  $L_{\varphi_{\alpha(\delta)}, \varphi^\dagger}$  such that

$$D_{\alpha J}(\varphi_{\alpha(\delta)}, \varphi^\dagger) \leq \alpha \kappa^2 |\Omega|^2 \mathcal{K}(\varphi_{\alpha(\delta)}) L_{\varphi_{\alpha(\delta)}, \varphi^\dagger} \|\varphi_{\alpha(\delta)} - \varphi^\dagger\|_2 + \delta \|\mathcal{T}\| \|\varphi_{\alpha(\delta)} - \varphi^\dagger\|_2.$$

By the given choice of regularization parameter rule (4.11), it is concluded that

$$D_{\alpha J}(\varphi_{\alpha(\delta)}, \varphi^\dagger) \leq \delta \|\varphi_{\alpha(\delta)} - \varphi^\dagger\|_2 + \delta \|\mathcal{T}^*\| \|\varphi_{\alpha(\delta)} - \varphi^\dagger\|_2. \quad (4.12)$$

Hence, by 2-convexity in (2.7) and strong convexity of the smoothed TV penalty (see Thrm 4.1),

$$\|\varphi_{\alpha(\delta)} - \varphi^\dagger\|_2 \leq \delta \left( \frac{1}{2} + \|\mathcal{T}^*\| \right) \rightarrow 0, \text{ as } \alpha(\delta) \rightarrow 0 \text{ whilst } \delta \rightarrow 0. \quad (4.13)$$

□

**COROLLARY 4.6** (Final discrepancy estimation for the smoothed-TV regularization). *Under the same assumptions of Theorem 4.5, if the regularization parameter is chosen with respect to the rule*

$$\alpha(\delta) \leq \delta \frac{(\mathcal{K}(\varphi_{\alpha(\delta)}))^{-1}}{2\kappa^2 |\Omega|^2 L_{\varphi_{\alpha(\delta)}, \varphi^\dagger}}, \quad (4.14)$$

for  $\delta \in (0, 1)$  sufficiently small, then by (4.6),

$$\|\mathcal{T}\varphi_{\alpha(\delta)} - f^\delta\|_{\mathcal{L}^2} \leq \delta \left( \frac{3}{2} + \|\mathcal{T}^*\| \right)^{1/2}. \quad (4.15)$$

*Proof.* In the proof of Theorem 4.5, we have observed

$$\|\mathcal{T}\varphi_{\alpha(\delta)} - f^\delta\|_{\mathcal{L}^2}^2 \leq \delta^2 + 2\alpha \kappa^2 |\Omega|^2 \mathcal{K}(\varphi_{\alpha(\delta)}) L_{\varphi_{\alpha(\delta)}, \varphi^\dagger} \|\varphi_{\alpha(\delta)} - \varphi^\dagger\|_2. \quad (4.16)$$

After plugging (4.14) in,

$$\|\mathcal{T}\varphi_{\alpha(\delta)} - f^\delta\|_{\mathcal{L}^2}^2 \leq \delta^2 + \delta\|\varphi_{\alpha(\delta)} - \varphi^\dagger\|_2. \quad (4.17)$$

Recall the rate of the convergence for  $\|\varphi_{\alpha(\delta)} - \varphi^\dagger\|_2$  when the choice of regularization parameter fulfils the condition (4.14), which is

$$\|\mathcal{T}\varphi_{\alpha(\delta)} - f^\delta\|_{\mathcal{L}^2}^2 \leq \delta^2 + \delta^2 \left( \frac{1}{2} + \|\mathcal{T}^*\| \right) = \delta^2 \left( \frac{3}{2} + \|\mathcal{T}^*\| \right).$$

This yields the result after taking the square root of both sides.

□

## Conclusion and Further Discussion

Hölder continuous functions in the application arises in the field of scattering theory, see for details [12, Section 8.2] and a recent work [23, Lemma 2.2]. In this work, we have explored the regularity properties of the smoothed-TV regularization for such functions. The scientific reason why smoothed-TV regularization has been chosen for such a study has been established in Theorem 3.3.

In Theorem 3.4, we have proved that  $TV$  of a function  $\varphi$  can be bounded by its Lipschitz constant  $\kappa$  which is the case of  $\gamma = 1$ . However, it is still an open question to be able to show that

$$TV(\varphi) \leq C(\Omega)[\varphi]_{\mathcal{C}^{0,\gamma}(\Omega)}$$

where  $C(\Omega)$  is a constant depending on the compact domain  $\Omega \subset \mathbb{R}^3$  and  $\gamma = 1/4$  due to Morrey's inequality, see Theorem 3.1. Then a new compact embedding theorem between the spaces  $BV(\Omega)$  and  $\mathcal{C}^{0,\gamma}(\Omega)$  can be established. On the other hand, compact embedding amongst the Hölder spaces with the different orders has already been proven, [12, Theorem 3.2].

Speaking about proving that smoothed TV regularization is another admissible regularization strategy, we have intentionally taken into account that the forward operator is compact. The reason behind that can be explained as follows; Application and analysis of the method has been widely carried out in the communities of inverse problem and optimization, [1, 3, 4, 8, 9, 10, 14, 15, 31]. It is well-known that the efficient result of TV (or smoothed TV) regularization usually comes from image processing where the compact operator is mostly considered to be identity operator, *i.e.*,  $\mathcal{T} = \mathcal{I}$ . Lagged diffusivity fixed point iteration is the easiest algorithm in order to approximate the solution for the problem (1.1), [10, 30, 31]. The convergence of this algorithm has been shown only for the case of  $\mathcal{T} = \mathcal{I}$ , [2, 10]. Following the same steps in the regarding works, we also define the following continuous nonlinear transformation

$$\mathcal{P}(\varphi_{\alpha(\delta)}) := \left( -\alpha(\delta)\nabla^* \cdot \left( \frac{\nabla}{(\beta + |\nabla\varphi_{\alpha(\delta)}|^2)^{1/2}} \right) + \mathcal{T}^*\mathcal{T} \right). \quad (4.18)$$

According to the regarding works, the algorithm is convergent in the condition of  $\lambda_{\min}(\mathcal{P}(\varphi_{\varphi_{\alpha(\delta)}})) \geq \sigma(\mathcal{T}^*\mathcal{T}) \geq 1$ . Obviously, this can not hold for us since our forward operator  $\mathcal{T}$  is compact. A tomographic application of total variation regularization with some compact forward operator has been recently studied in [21].

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